to make the principle appear of real value on account of its applications. That such problems exist has been shown, but the teacher can do a great service by adding to the supply existent.

6. A high school view of equations. Almost invariably, the student who has completed merely the algebra offered in our secondary schools, regards the equation solely as an equality containing unknowns to be found. To find these unknowns, the student has learned to perform certain mechanical processes on the members of the equation — these processes being suggested by the simple operations of arithmetic and not really established for the more general numbers of algebra. The validity of the processes under any conditions is hardly ever called into question in the secondary algebra. Whether substantial improvement in results bearing on this point are to be expected from secondary schools in the near future is beyond the purpose of this paper. But, at present, surely, the condition obtains that in so far as the equation is an object of thought for the entering freshman, he regards it almost solely with reference to finding the unknowns it contains.

7. A second view of equations. In college algebra, the chief emphasis should be placed on the equation as a *condition* to be satisfied rather than as something containing unknowns to be found. To show that equations are not thus regarded by students entering college, and that this fact leads to some difficulties, let me say that I find every year in discussing the factor theorem that some of the superior students argue as follows:

"If a is a root of $f(x)=0$, this means that the unknown x is a, and to say that $f(x)$ is divisible by $x-a$ is equivalent to saying that zero is a divisor of $f(a)$. But this is contrary to the fundamental statement that division by zero is excluded from mathematical operations."

To lead the student to correct this fallacy, and to substitute for it correct views on such points is one of the difficult tasks of the instructor in college algebra.

8. Idea of functionality. In order to appreciate fully and deeply the significance of satisfying even simple equations such as $y=3x+4$ and $y=4x-6$, the graph is almost indispensable. With the graph comes the idea of functionality; and, in my opinion, a treatment of the graphs of linear and quadratic functions should precede the review of the linear and quadratic equations in this course.

To get some idea of the recent tendency to appeal to geometrical representations in college algebra, I have just examined eight text books in college algebra that I happen to have on hand — four of these books have been written within the past five years, and four others from ten to nineteen years ago. I find that the newer set of books are adorned with curves and figures to the extent of from twenty-four to thirty-nine cuts while the older books contain from none to nineteen cuts — most of the latter being given to represent complex numbers. While I recognize that the character of the text is not, in general, identical with the course, I think the change in the

character of text books is a significant measure of the tendency to make graphs and the idea of functionality more prominent than formerly.

9. Determinants. In expressing solutions of linear equations, and in placing conditions on coefficients of various equations to correspond to certain facts in regard to the roots, the determinant notation is such an important mechanical aid in giving elegance of form to the work that the elements of determinants should be presented for use in this course and in analytical geometry, rather than as an isolated subject almost entirely separated from the equation.

10. Extensions of the number concept. In seeking numbers to satisfy equations, there is the opportunity to interest the student in extensions of the number concept, and to treat in a very elementary way irrational and complex numbers. The writer has been much impressed by the interest of his students in making extensions of the number concept appear necessary to meet the demands of the equation. This method of extending the number concept may, of course, give a wrong view unless it is pointed out that some irrational numbers, such as π for example, are not roots of an equation with rational coefficients.

11. Theory of equations. In the time at our disposal, we can perhaps expect only that the solution for real roots of special numerical equations be reached as a goal in the theory of equations. This problem is not only of value for its direct applications, but also because it brings before the student a process of successive approximation of much importance as an illustration of general methods of successive approximation. However, the points to be kept most prominently before the student in the theory of equations are that we are regarding $f(x)=0$ as a relation to be satisfied, and $f(x)$ as a function whose changes in value concern us.

11. Logarithms. In the plotting of graphs of functions, expressions of sufficient complexity should be given to bring the student to seek the best methods of numerical evaluation. This calls for the use of logarithms, which, it is generally hoped, are used in calculations in the secondary course, but are often not used in such a course.

12. Limits and infinite series. There seems to exist considerable difference of opinion among instructors of college mathematics as to the advisability of presenting the elements of the theory of limits and infinite series in a freshman course. If the subject of infinite series is to be included in the course, it can be very effectively presented with emphasis on the fact that the first n terms of a series is a function which changes as n takes values 1, 2, 3, 4, ..., and that it is the limit of this function as n grows beyond any fixed bound in which we are interested. In this way, the subject of infinite series is referred to one of the unifying elements of the course.

To write the generating function from a few terms of a series is a most valuable exercise in leading students to grasp what may be called algebraic form. Herein is the recognition of algebraic law from particular

cases. Even the simple direct process of expanding $\sum_{n=1}^{n=\infty} u_n$, where u_n is some function of n such as $\frac{1}{(2n-1)\,!}$, forms a sort of prelude to the use of the summation sign in the calculus.

The value of the subject of infinite series does not therefore rest solely either on the practical ground that the student needs to test the convergence of special series that arise in analysis, nor on the ground that it is a suitable field for the growth of the limit concept.

The writer does not find himself in agreement with those who hold that it is a more natural order of presentation to defer the study of infinite series until late in the course in calculus. He questions whether this position is tenable, especially if series are presented with a considerable use of geometric representation. That is to say, if the sums of n terms of a few carefully selected series are plotted as ordinates to correspond to n=1, 2, 3, ... used as abscissas, so that the function idea is prominent, the process of approach to a limit makes a valuable appeal to the student's common sense. While the answer to the question of including this subject in a freshman course may be changed by the amount of time allotted to the course in calculus, it is the opinion of the writer that a good deal is gained by presenting to freshmen students in considerable detail, the elementary tests for the convergence and divergence of series, if fewer than five hours per week through a school year are given to calculus.

Wherever convergence of series is studied, the treatment should be marked by precision of statements. The arguments should be in harmony with the unifying principles of this course in algebra; that is, assumptions on which proofs are based should be explicitly stated.

13. Summary. In this paper, the writer has aimed to consider especially certain subjects whose place in college algebra is sometimes called into question, and to direct attention to points which should be particularly emphasized.

It seems that while the course in college algebra should be unified, to some extent, by logical considerations, it should be unified in method of presentation by much reference to 1) the equation as a relation to be satisfied, 2) the function as a variable whose changes in value are to be traced.

It is by referring much to these central and closely related ideas, and by the introduction of more practical problems (§5) that we may well expect to improve our teaching of college algebra.

ON r-FOLD SYMMETRY OF PLANE ALGEBRAIC CURVES.*

By R. D. CARMICHAEL, Princeton, N. J.

If a plane curve is revolved about a point in its own plane through an angle of $360°/r$ and if it then coincides with its former position, it is said to have r-fold symmetry with respect to the point; and the point is called the center of r-fold symmetry. The object of this paper is to ascertain the analytical conditions which are necessary and sufficient to the existence of r-fold symmetry and to examine into the geometric properties of the curves in certain special cases. In a previous note† I have given a classification of plane algebraic curves having four-fold symmetry about a point, and this has been followed‡ by a paper on the geometric properties of quartic curves possessing such four-fold symmetry.

In the present discussion we shall confine ourselves to plain algebraic loci which are such that no locus is composed entirely of isolated points or of straight lines; in other words, every locus considered will be assumed to have at least one part which is *continuous* and *curved*. And this assumption is made throughout without further statement.

Evidently the circle is a curve of infinite-fold symmetry. It is clear that the condition of infinite-fold symmetry with respect to the origin is that the polar equation shall be independent of the vectorial angle; that is, the locus in this case is a circle or a system of concentric circles with center at the center of infinite-fold symmetry. Therefore it will be sufficient in what follows to confine our attention to the cases in which r is finite.

1. Separation into two classes. Let n be the order of a curve of r-fold symmetry and let it be referred to rectangular cartesian coordinates with origin at the center of r-fold symmetry. Take the equation in the form

$$\sum a_{ts}x^t y^s = 0, \tag{1}$$

where a_{ts} is a real constant for every t and s and where t and s each range over the values 0, 1, 2, ..., n subject to the condition

$$t+s \leqq n.$$

Certain relations must exist among the coefficients a. These are now to be found.

If we transform equation (1) by the substitution

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†*Annals of Mathematics*, Vol. 9, No. 2, pp. 53-56.
‡*Annals of Mathematics*, Vol. 10, No. 2, pp. 81-87.

$$x=\rho\cos\theta,\quad y=\rho\sin\theta,$$

we have

$$(2)\qquad \sum_{\nu=0}^{\nu=n}\rho^{\nu}\sum_{t,s=0}^{t,s=\nu}a_{ts}\cos^{t}\theta\sin^{s}\theta=0,\quad (t+s=\nu).$$

Putting $\theta=\theta_1$ and $\theta=\theta_1+\alpha\phi$ where

$$\phi=360^\circ/r \text{ and } \alpha=\text{an integer},$$

we have the following equations:

$$(3)\qquad \sum_{\nu=0}^{\nu=n}\rho^{\nu}\sum_{t,s=0}^{t,s=\nu}a_{ts}\cos^{t}\theta_1\sin^{s}\theta_1=0,\quad (t+s=\nu),$$

$$(4)\qquad \sum_{\nu=0}^{\nu=n}\rho^{\nu}\sum_{t,s=0}^{t,s=\nu}a_{ts}\cos^{t}(\theta_1+\alpha\phi)\sin^{s}(\theta+\alpha\phi)=0,\quad (t+s=\nu).$$

From the existence of the defined r-fold symmetry it follows that equations (3) and (4) must yield by solution the same values of ρ. Therefore the coefficients can differ only by a constant factor m_α; that is,

$$(5)\qquad \sum_{t,s=0}^{t;s=\nu}a_{ts}\cos^{t}\theta_1\sin^{s}\theta_1=m_\alpha\sum_{t,s=0}^{t,s=\nu}a_{ts}\cos^{t}(\theta_1+\alpha\phi)\sin^{s}(\theta_1+\alpha\phi),\quad (t+s=\nu).$$

Equation (5) must hold for each value of ν from 0 to n and for each value of α from 1 to r, a different equation being formed for every case. Then (5) yields $r(n+1)$ equations which must all be satisfied for every possible value of θ_1. It is clear that the existence of this system of equations is both necessary and sufficient to the existence of the defined symmetry.

We shall now evaluate the constants m_α. If we take $\alpha=1$ and multiply equation (4) by m_1 (which evidently cannot be zero) it follows that the result is identical with equation (3). Hence a second addition of ϕ to the vectorial angle would necessitate a second multiplication of the coefficients by m_1; that is, two multiplications by m_1 produces m_2; or

$$m_2=m_1^2.$$

Continuing the additions of ϕ to the vectorial angle, we have

$$m_2=m_1^2,\quad m_3=m_1^3,\quad \ldots,\quad m_r=m_1^r.$$

At the rth addition of ϕ to the vectorial angle the two members of (5) become identical except as to the presence of the factor m_r in the second member; and therefore m_r must equal 1. For, if not, we must have

$$\Sigma\, a_{ts} \cos^t \theta_1 \sin^s \theta_1 = 0, \quad (t+s=\nu),$$

for an unlimited number of values of θ_1 each less than 2π; and this is evidently impossible. Since $m_r = m_1{}^r$ and $m_r = 1$, we have $m_1{}^r = 1$. It is evident from (5) that m_1 is a real quantity. Therefore

(6) $\quad m_1 = +1$, when r is odd;

(7) $\quad m_1 = \pm 1$, when r is even;

(8) $\quad m_\alpha = m_1{}^\alpha$, in every case.

In order to find the necessary and sufficient relations among the coefficients a we proceed as follows. For $m_1 = +1$, $m_1 = -1$, equation (5) takes the respective forms:

$$(9)\quad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t\theta_1 \sin^s\theta_1 = \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t(\theta_1+\alpha\phi) \sin^s(\theta_1+\alpha\phi), \ (t+s=\nu), \ [A],$$

$$(10)\quad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t\theta_1 \sin^s\theta_1 = (-1)^\alpha \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t(\theta_1+\alpha\phi) \sin^s(\theta_1+\alpha\phi), \ (t+s=\nu), \ [B],$$

where ν ranges over all the values 0, 1, 2, ..., n, different equations being formed for each value of ν. Equation (9) alone holds when r is odd; when r is even both equations (9) and (10) may hold. Evidently these equations are necessary and sufficient to the existence of r-fold symmetry; that is, for odd-fold symmetry we must be able to satisfy (9); for even-fold symmetry, either (9) or (10) or both. In case this condition cannot be satisfied for given r and n, we are to conclude that r-fold symmetry does not exist for curves of such degree n. As a case in point, we have the theorem: Four-fold symmetry does not exist for curves of odd degree.*

We shall say that curves which satisfy equations (9) and (10) are of class A and B, respectively. In class A there will be found loci of both odd- and even-fold symmetry; in class B will be found loci of only even-fold symmetry.

Obviously, if the equation of a curve referred to rectangular axes has only terms of even degree or only terms of odd degree, the curve has two-fold symmetry; for in either case, if α, β is a point on the curve, so is $-\alpha$,

Annals of Mathematics, Vol. 9, No. 2, p. 55.

$-\beta$. Conversely, if the origin of rectangular coordinates is taken at the center of two-fold symmetry, the equation must evidently have one of the forms indicated. If each term is of even degree, it is obvious that the curve belongs to class A; while if each term is of odd degree, the curve belongs to class B.

2. Determination of constants a_{ts} for class A. Equation (9) indicates that the real function

$$F \equiv \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1, \quad (t+s=\nu), \tag{11}$$

is periodic with the real period $\phi = 2\pi/r$. But every real function of a single variable θ_1 with the real period $2\pi/r$ can be expanded in a Fourier series in the general form

$$\sum_{i=0}^{\infty} c_i \cos ir\,\theta_1 + \sum_{i=1}^{\infty} y_i \sin ir\,\theta_1. \tag{12}$$

If $\cos ir\,\theta_1$ and $\sin ir\,\theta_1$ are expanded in terms of $\sin\theta_1$ and $\cos\theta_1$ the results are homogeneous of order ir in $\sin\theta_1$, $\cos\theta_1$; moreover, the coefficients c_i and y_i do not belong to terms alike in $\sin\theta_1$ and $\cos\theta_1$, and therefore cannot annul each other. Hence, if the expression in (12) is to be identical with F, $ir \leq \nu$. If $ir = \nu - 2j$, the corresponding part of (12) when expanded in terms of $\cos\theta_1$, $\sin\theta_1$ is of degree $\nu - 2j$; but it becomes of degree ν through multiplication by the unit factor $(\cos^2\theta_1 + \sin^2\theta_1)^j$. Evidently ir cannot differ from ν by an odd number. Hence, as a result we have

$$\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{i=0}^{i} c_i \cos ir\,\theta_1 + \sum_{i=1}^{i} y_i \sin ir\,\theta_1, \tag{13}$$

where $t+s=\nu$ and ir is always positive and has as its values some or all of the positive numbers of the series ν, $\nu-2$, $\nu-4$, ...

Now, if ν ranges from 0 to n, the preceding result enables us to determine readily the values of all of the coefficients a_{ts} in terms of a suitable number of them selected as independent constants. Substituting these values in (1) we obtain the most general form of the equation of the nth degree locus possessing r-fold symmetry and belonging to class A as defined above. Such equations, for several values of r and n, are written out below in their most general form.*

*For curves of four-fold symmetry see my previous papers already referred to. Curves of two-fold symmetry are disposed of at the close of section 1 of this paper.

SOME CURVES OF CLASS A OF 3-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2(x^2+y^2) = 0.$$

$$F_3 \equiv F_2 + c_3 x^3 + c_4 y^3 - 3c_4 x^2 y - 3c_3 x y^2 = 0.$$

$$F_4 \equiv F_3 + c_5(x^2+y^2)^2 = 0.$$

$$F_5 \equiv F_4 + c_6 x^5 + c_7 y^5 - 3c_7 x^4 y - 3c_6 x y^4 - 2c_6 x^3 y^2 - 2c_7 x^2 y^3 = 0.$$

$$F_6 \equiv F_5 + c_8(x^6 - y^6) + 6c_9 xy(x^4+y^4) - 15c_8 x^2 y^2 (x^2 - y^2) - 20c_9 x^3 y^3 + c_{10}(x^2+y^2)^3 = 0.$$

$$F_7 \equiv F_6 + (x^2+y^2)(c_{11} x^5 + c_{12} y^5 - 3c_{12} x^4 y - 2c_{11} x^3 y^2 - 2c_{12} x^2 y^3) = 0.$$

SOME CURVES OF CLASS A OF 5-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2(x^2+y^2) = 0.$$

$$F_4 \equiv F_2 + c_3(x^2+y^2)^2 = 0.$$

$$F_5 \equiv F_4 + c_4 x^5 + c_5 y^5 + 5c_5 x^4 y + 5c_4 x y^4 - 10c_4 x^3 y^2 - 10c_5 x^2 y^3 = 0.$$

$$F_6 \equiv F_5 + c_6(x^2+y^2)^3 = 0.$$

$$F_7 \equiv F_6 + (x^2+y^2)(c_7 x^5 + c_8 y^5 + 5c_8 x^4 y + 5c_7 x y^4 - 10c_7 x^3 y^2 - 10c_8 x^2 y^3 = 0.$$

SOME CURVES OF CLASS A OF 6-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2(x^2+y^2) = 0.$$

$$F_4 \equiv F_2 + c_3(x^2+y^2)^2 = 0.$$

$$F_6 \equiv F_4 + c_4(x^2+y^2)^3 + c_5(x^6 - y^6) + 6c_6 xy(x^4+y^4) - 15c_5 x^2 y^2 (x^2 - y^2) - 20c_6 x^3 y^3 = 0.$$

SOME CURVES OF CLASS A OF 7-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2(x^2+y^2) = 0.$$

$$F_4 \equiv F_2 + c_3(x^2+y^2)^2 = 0.$$

$$F_6 \equiv F_4 + c_4(x^2+y^2)^3 = 0.$$

$$F_7 \equiv F_6 + c_5 x^7 + c_6 y^7 - 7c_6 x^6 y - 7c_5 x y^6 - 21c_5 x^5 y^2 - 21c_6 x^2 y^5 + 35c_6 x^4 y^3 + 35c_5 x^3 y^4 = 0.$$

3. Determination of the constants a_{is} for class B. For class B we have seen that r is even. It may be shown that r-fold symmetry in class B is a special case of $\frac{1}{2}r$-fold symmetry in class A. For if ϕ is the angle through which the r-fold (r even) symmetrical curve of class B must be turned in order to coincide with its original position, 2ϕ is the angle through which the $\frac{1}{2}r$-fold symmetrical curve of class A must be turned that it may coincide with its original position. But the former still coincides with its first posi-

tion after being turned through an angle of 2ϕ. Hence, since r must be even for curves of class B, r-fold symmetry of class B is a special case of $\frac{1}{2}r$-fold symmetry of class A.

For $\frac{1}{2}r$-fold symmetry equation (13) becomes

$$(14) \qquad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{i=0}^{i} c_i \cos\frac{ir}{2}\theta_1 + \sum_{i=1}^{i} y_i \sin\frac{ir}{2}\theta_1,$$

where $t+s=\nu$ and $ir/2$ is always positive and has as its values some or all the positive numbers of the series ν, $\nu-2$, $\nu-4$, ... This is a necessary condition for r-fold symmetry in class B. From (10) we may write

$$(15) \qquad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = -\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t (\theta_1+\phi) \sin^s (\theta_1+\phi), \quad (t+s=\nu).$$

Since the existence of equation (10) is a necessary and sufficient condition for r-fold symmetry of curves of class B, it is readily seen from the discussion in the preceding paragraph that the existence at the same time of equations (14) and (15) is also a necessary and sufficient condition for r-fold symmetry of curves of class B. This result enables one to determine the constants a_{ts} in terms of a suitable number of them chosen as independent constants.

But if r is twice an odd number, the constants may be more readily determined in the following manner: In the equation for the curve of class A of $\frac{1}{2}r$-fold symmetry, insert the condition for two-fold symmetry of class B; that is, let the equation consist only of terms of odd degree. In this way were found the equations for 6- and 10-fold symmetry given below.

SOME CURVES OF CLASS B OF 6-FOLD SYMMETRY.

$$F_3 \equiv c_1x^3+c_2y^3-3c_2x^2y-3c_1xy^2=0.$$

$$F_5 \equiv F_3+c_3x^5+c_4y^5-3c_4x^4y-3c_3xy^4-2c_3x^3y^2-2c_4x^2y^3=0.$$

$$F_7 \equiv F_5+(x^2+y^2)(c_5x^5+c_6y^5-3c_6x^4y-3c_5xy^4-2c_5x^3y^2-2c_6x^2y^3) + (x^2+y^2)^2(c_7x^3+c_8y^3-3c_8x^2y-3c_7xy^2)=0.$$

SOME CURVES OF CLASS B OF 10-FOLD SYMMETRY.

$$F_5 \equiv c_1x^5+c_2y^5+5c_2x^4y+5c_1xy^4-10c_1x^3y^2-10c_2x^2y^3=0.$$

$$F_7 \equiv F_5+(x^2+y^2)(c_3x^5+c_4y^5+5c_4x^4y+5c_3xy^4-10c_3x^3y^2 -10c_4x^2y^3)=0.$$

$$F_9 \equiv F_7+(x^2+y^2)^2(c_5x^5+c_6y^5+5c_6x^4y+5c_5xy^4-10c_5x^3y^2 -10c_6x^2y^3)=0.$$

A different method, however, is necessary for curves of 8-fold symmetry. In this case we must employ equations (14) and (15); or, what is the same thing, a corollary from my first paper on four-fold symmetry (already referred to) and equation (15): namely, the necessary and sufficient condition for four-fold symmetry of curves of class A is that every term in the equation shall be of even degree and that

$$a_{st}=(-1)^t a_{ts}.$$

(This result is not explicitly stated there, but is easily deduced as a corollary from the argument.) This enables us in the present case to write (15) in different form. We replace ϕ by its value 45°.

$$(16)\quad \sum_{t,\,s=0}^{t,\,s=\nu}[a_{ts}(\cos^t\theta_1\sin^s\theta_1+(-1)^t\cos^s\theta_1\sin^t\theta_1)]$$

$$=-\sum_{t,\,s=0}^{t,\,s=\nu}\{a_{ts}[\cos^t(\theta_1+45^\circ)\sin^s(\theta_1+45^\circ)+(-1)^t\cos^s(\theta_1+45^\circ)\sin^t(\theta_1+45^\circ)]\},$$

where $t+s=\nu$, ν being an even number and $t \gtrless s$. It follows that the existence of equation (16) is the necessary and sufficient condition for 8-fold symmetrical curves of class B. By its aid one may determine the equations of 8-fold symmetrical loci.

4. A simplification in constructing the equations in general. If

$$r=p_1^{a_1}p_2^{a_2}\dots p_k^{a_k},$$

where $p_1, p_2, \dots, p_k$ are different primes, we may evidently proceed as follows to construct the equations of nth degree loci possessing r-fold symmetry:

Construct the equations of nth degree loci possessing symmetry of class A and of orders $p_1^{a_1}, p_2^{a_2}, \dots, p_k^{a_k}$, respectively. From these construct the most general equation in which the coefficients obey all the limitations imposed in the several equations separately. The result is the most general form of the equation of class A. Proceed similarly for class B.

5. An example. As an illustrative example consider a special case of the seventh degree curve of class B of 10-fold symmetry in the table above. Let $c_1=c_4=0$, $c_2\neq 0$, $c_3\neq 0$. Then the equation is of the form:

$$(17)\quad a_1(5x^4y-10x^2y^3+y^5)+a_2(x^2+y^2)(x^5-10x^3y^2+5xy^4)=0,\ a_1\neq 0,\ a_2\neq 0.$$

Transforming to polar coordinates by the substitution $x=\rho\cos\theta$, $y=\rho\sin\theta$, and substituting $\cos 5\theta$ and $\sin 5\theta$ for $\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ and $5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$, respectively, the equation becomes

$$a_1\rho^5\sin 5\theta+a_2\rho^7\cos 5\theta=0.$$

Evidently this may be replaced by the two equations

$$\rho^5=0,$$

(18) $$\rho^2=a\tan 5\theta,$$

where $$a=-a_1/a_2.$$

To the former of these two equations corresponds only the origin. But this point is on the locus of the other equation. Hence, so far as plotting the curve is concerned, equation (17) may be replaced by equation (18). Evidently, it consists of five branches, alike except for position. Each branch passes through the origin and has in itself two-fold symmetry with respect to the origin. Moreover, the origin is obviously a point of inflection for each branch, and there are thus five (but only five) points of inflection at the origin. Now, since the curve possesses 10-fold symmetry, singularities not at the origin can enter only by tens. Hence the number of points of inflection is an odd multiple of 5. It is easy to see that there is no cusp at the origin. Hence cusps enter only in tens, if at all.

For the further discussion of singularities we require the following Plücker equations, which are written in the ordinary notation:

(19) $$m=n(n-1)-(2\delta+3\rho),$$

(20) $$n=m(m-1)-(2\tau+3\iota),$$

(21) $$\iota=3n(n-2)-(6\delta+8\rho),$$

(22) $$\rho=3m(m-2)-(6\tau+8\iota).$$

We now have $n=7$, ι=odd multiple of 5, ρ=multiple of 10, or zero. Then from (21) it may be seen that $6\delta+8\rho$ must be an even multiple of 5; that is, a multiple of 10. But ρ is a multiple of 10, or zero; hence δ is a multiple of 5. It is obvious from (20) that $m\geqq 4$; hence from (19) it follows that either δ or ρ is zero; and therefore $\rho=0$, since the curve under consideration has double points at the origin. Now the locus is of the seventh degree and cannot have as many as ten coincident points; hence, since δ is a multiple of 5, the number of double points at the origin is 5. Therefore, from (19) it follows that $\delta=5$ or 15, since singularities not at the origin enter only by tens. We shall now determine which of these is the true value.

Suppose that there is a double point not at the origin; and let it be at a distance d from the origin. Then there must be ten such double points at a distance d from the origin. Pass through them a circle with radius d

and center at the origin. Since each double point counts as two points the circle cuts the septic curve in 20 points. But this is impossible, hence there is no double point except at the origin. Therefore $\delta=5$. Then from Plücker's equations: $m=32$, $\iota=75$, $\tau=380$. Hence the curve

$$5x^4y-10x^2y^3+y^5+c(x^2+y^2)(x^5-10x^3y+5xy^4)=0,\ c\neq 0.$$

is of class 32, is non-cuspidal, and has five double points at the origin, 75 points of inflection of which five are at the origin, and 380 double tangents. It is obvious that not all the singularities are real.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n=\infty} x^{1/(2n-1)},\ n=1, 2, \ldots;\quad f(x)=\lim_{n=\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

No solution of this problem has been received.

331. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Extract the square root of $21+6\sqrt{2}+2\sqrt{21}-6\sqrt{3}-6\sqrt{7}-2\sqrt{6}-2\sqrt{14}$ and also of $4\sqrt{2}+2\sqrt{6}-9-4\sqrt{3}$.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y., and J. SCHEFFER, A.M., Hagerstown, Md.

(a) Assume the root to be of the form

$$a\sqrt{2}+b\sqrt{3}+c\sqrt{7}+d.$$

Squaring and comparing coefficients, we have

$$ab=-1,\ ac=-1,\ ad=3,\ bc=1,\ bd=-3,\ cd=-3,$$
$$2a^2+3b^2+7c^2+d^2=21.$$

Whence $a=-1$, $b=1$, $c=1$, $d=-3$, and the root is $\sqrt{3}-\sqrt{2}+\sqrt{7}-3$.

For second expression, (b), assume root of form $\sqrt{-1}(a+b\sqrt{3}+c\sqrt{2})$. Squaring and comparing coefficients, we have,

$$ab=2,\ ac=-2,\ bc=-1,\ a^2+3b^2+2c^2=9.$$

Whence $a=2$, $b=1$, $c=-1$, and the root is $\sqrt{-1}(2+\sqrt{3}-\sqrt{2})$.

Solved similarly by V. M. Spunar and Levi S. Shively.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The method used here was suggested to me by Mr. Githens in a solution for cube root. It is considerably simpler than methods ordinarily used.

(a) Let $A=[21+6\sqrt{2}+2\sqrt{21}-(6\sqrt{3}+6\sqrt{7}+2\sqrt{6}+2\sqrt{14})]^{\frac{1}{3}}$
$=\sqrt{(m-n)}$.

Then $\sqrt{[(m-n)(m+n)]}=[157+12\sqrt{2}-(4\sqrt{21}+24\sqrt{42})]^{\frac{1}{3}}=\sqrt{(p-q)}$.
$\sqrt{[(p-q)(p+q)]}=(409-264\sqrt{2})^{\frac{1}{3}}=\sqrt{(r-s)}$.
$\sqrt{[(r-s)(r+s)]}=167$.

Let $\sqrt{(r-s)}=\sqrt{(x+167)}-\sqrt{x}$. Squaring both members and equating rational terms, $2x+167=409$, or $x=121$.

$\therefore \sqrt{(r-s)}=12\sqrt{2}-11$.
Let $\sqrt{(p-q)}=\sqrt{(x+12\sqrt{2}-11)}-\sqrt{x}$.
Then $2x+12\sqrt{2}-11=157+12\sqrt{2}$, or $x=84$.
$\therefore \sqrt{(p-q)}=\sqrt{(73+12\sqrt{2})}-2\sqrt{21}$.
$\sqrt{[(73+12\sqrt{2})(73-12\sqrt{2})]}=71$.
Hence $\sqrt{(73+12\sqrt{2})}=\sqrt{(x+71)}+\sqrt{x}$.
From this, $x=1$ and $\sqrt{(73+12\sqrt{2})}=6\sqrt{2}+1$.
$\sqrt{(p-q)}=6\sqrt{2}+1-2\sqrt{21}$.
Let $\sqrt{(m-n)}=\sqrt{(x+6\sqrt{2}+1-2\sqrt{21})}-\sqrt{x}$.
Then $2x+6\sqrt{2}+1-2\sqrt{21}=21+6\sqrt{2}+2\sqrt{21}$.
$\therefore x=10+2\sqrt{21}$. $\therefore \sqrt{(m-n)}=\sqrt{(11+6\sqrt{2})}-\sqrt{(10+2\sqrt{21})}$.

$\sqrt{[(11+6\sqrt{2})(11-6\sqrt{2})]}=7$, and $\sqrt{(11+6\sqrt{2})}=\sqrt{(x+7)}+\sqrt{x}$... (1).
$\sqrt{[(10+2\sqrt{21})(10-2\sqrt{21})]}=4$, and $\sqrt{(10+2\sqrt{21})}=\sqrt{(x+4)}+\sqrt{x}$... (2).

From (1), $x=2$; from (2), $x=3$.
$\therefore \sqrt{(11+6\sqrt{2})}=3+\sqrt{2}$, $\sqrt{(10+2\sqrt{21})}=\sqrt{7}+\sqrt{3}$.
$\therefore \sqrt{(m-n)}=3+\sqrt{2}-\sqrt{7}-\sqrt{3}$.

(b) Let $(-9-4\sqrt{3}+4\sqrt{2}+2\sqrt{6})^{\frac{1}{2}} = \sqrt{(-m+n)}$.

$\sqrt{[(-m+n)(-m-n)]}=\sqrt{[(-m)^2-n^2]} = \sqrt{[73+40\sqrt{3}]}=\sqrt{[p+q]}$.

$\sqrt{[(p+q)(p-q)]}=23$.

Let $\sqrt{(p+q)}=\sqrt{(x+23)}+\sqrt{x}$. Then $2x+23=73$, or $x=25$.

$\therefore \sqrt{(p+q)}=4\sqrt{3}+5$.

Let $\sqrt{(n-m)}=-\sqrt{(x+4\sqrt{3}+5)}+\sqrt{x}$. Then $2x+4\sqrt{3}+5=-9-4\sqrt{3}$, or $x=-7-4\sqrt{3}$).

$\therefore \sqrt{(n-m)}=\sqrt{(-7-4\sqrt{3})}-\sqrt{-2}$.

$\sqrt{[(-7-4\sqrt{3})(-7+4\sqrt{3})]}=1$.

Let $\sqrt{(-7-4\sqrt{3})}=\sqrt{(x+1)}+\sqrt{x}$; then $2x+1=-7$, $x=-4$.

$\therefore \sqrt{(-7-4\sqrt{3})}=\sqrt{(-3)}+2\sqrt{(-1)}$.

$\therefore \sqrt{(n-m)}=\sqrt{(-3)}+2\sqrt{(-1)}-\sqrt{(-2)}$.

Also solved by G. I. Hopkins and A. H. Holmes.

332. Proposed by C. N. SCHMALL, New York City.

Solve the quadratic, $x^2+ax+b=0$, without completing the square.

Solution by ARTEMAS MARTIN, LL. D., Washington, D. C.

Assume $y-\frac{1}{2}a=x$, and substitute in the given quadratic and it becomes

$$(y-\tfrac{1}{2}a)^2+a(y-\tfrac{1}{2}a)+b=0, \text{ or } y^2-\tfrac{1}{4}a^2+b=0;$$

whence $y=\pm\sqrt{(\frac{1}{4}a^2-b)}$ and $x=\pm\sqrt{\frac{1}{4}a^2-b)}-\frac{1}{2}a$.

See *Mathematical Magazine*, Vol. I, No. 9 (January, 1884), p. 146.

Solved similarly by V. M. Spunar, Levi S. Shively and the Proposer.

Professor Hopkins, in his solution, made use of the principle that the sum of the roots is equal to the coefficient of x with sign changed, and the product of the roots is equal to the final term.

S. Lefschetz sent in solutions of 327 and 328 too late for credit in last issue.

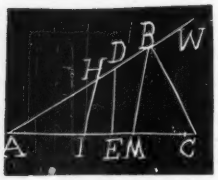
GEOMETRY.

356. Proposed by G. I. HOPKINS, Manchester, N. H.

Required to construct a triangle having given, base, vertical angle, and difference of other two sides.

I. Solution by J. M. ARNOLD, Crompton, R. I.

Let x=the longer side, then $x-d$=the shorter side. Let A=the vertical angle. Then



$$x^2-(x\cos A)^2=(x-d)^2-(b-x\cos A)^2$$

which gives

$$(2b\cos A-2d)x=b^2-d^2.$$

Dividing by 4 and putting in the form of a proportion

$$(\tfrac{1}{2}b\cos A-\tfrac{1}{2}d) : \frac{b-d}{2}=\frac{b+d}{2} : x.$$

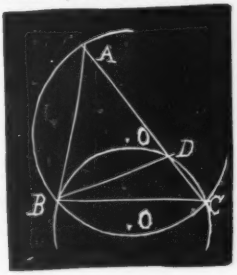
Construction. Draw AC equal to b, draw the indefinite line AW, making an angle at A equal to the given angle. On AW lay off AD equal to $\frac{1}{2}b$, and draw DE perpendicular to AC. From E lay off EI equal to $-\frac{1}{2}d$. Then will AI equal $\frac{1}{2}b\cos A-\frac{1}{2}d$.

On AW lay off AH equal to $\frac{b-d}{2}$ and draw HI. On AC lay off AM equal to $\frac{b+d}{2}$, and through M draw a line parallel to HI meeting AW at B. Join BC. Then will ABC be the triangle required.

Similarly solved by G. B. M. Zerr.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

On the given base, BC, as a chord, describe a circle O, containing the segment whose angle contains the angle$=90°+\frac{1}{2}A$, A being the given vertical angle, and also a circle O', the segment of which contains the angle A. Make BD=given difference of sides; extend BD to A, where it cuts the circumference of circle O'. Draw AC; then ABC is the required triangle.



For, $\angle ADC=90°-\frac{1}{2}A$.

$\therefore \angle ACD=90°-\frac{1}{2}A$; $\therefore AD=AC$; $\therefore AB-AC=BD$=given difference, which proves construction.

Solved similarly C. N. Schmall and H. C. Feemster.

357. Proposed by E. R. HOYT, St. Louis, Mo.

A room is 30 feet long, 12 feet wide, and 12 feet high. At one end of the room, 3 feet from the floor, and midway from the sides, is a spider. At the other end, 9 feet from the floor, and midway from the sides, is a fly. Determine the shortest path by way of the floor, ends, sides, and ceiling, the spider can take to capture the fly.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

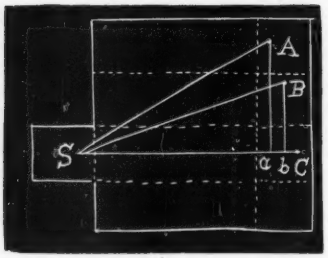
Suppose the six sides of the room spread out in one plane as in the figure, the floor being the second rectangle from the bottom, and let x=distance of spider from floor, $12-x$ the distance of the fly, $x<6$. There are three courses for the spider to take.

First, the route $SC=30+x+12-x=42$ feet... (1).

Second, the route $SB=\sqrt{[(Sb)^2+(Bb)^2]}=\sqrt{[(36+x)^2+(18-x)^2]}$... (2).

Third, the route $SA=\sqrt{[(Sa)^2+(Aa)^2]}=\sqrt{[(30+2x)^2+(24)^2]}$... (3).

Let $(30+2x)^2+576=(36+x)^2+(18-x)^2$.



Then $x^2+42x=72$, and $x=1.6495$ feet.

Let $(36+x)^2+(18-x)^2=(42)^2$.

Then $x^2+18x=72$, and $x=3.3693$ feet.

Then if $x<1.6495$ feet, the third route is shortest.

If $x>1.6495$ and <3.3693, the second route is shortest.

If $x>3.3693$, the first route is the shortest.

Since $x=3$, the spider takes the second route, and travels $\frac{1}{3}[(39)^2+(15)^2]=\frac{1}{3}[13^2+5^2)=\frac{1}{3}(194)<[\frac{1}{3}(196)=3\times14=42]$ feet.

Also solved by C. N. Schmall.

358. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Neb.

Cut four coplanar non-copunctual straight lines in a harmonic range.

I. Solution by the PROPOSER.

Let a, b, c, and d be the four lines, not meeting in a point. Let a and b meet at x. Draw two lines cutting a, b, and d at A, A_1, A_2 and A, B_1, B_2, respectively. Draw A_1B_2 and A_2B_1 meeting at C_1. Draw AC_1 cutting b and d at C and C_2. Draw XC_2, and draw XC_1 cutting C at D_2. Draw C_2D_2 cutting a and b at D and D_1. $DD_1D_2C_2$ is the required range.

For ACC_1C_2 is a harmonic range determined by the four-point $A_1B_1B_2A_2$. Hence, $X-ACC_1C_2$ is a harmonic pencil. So also is $X-DD_1D_2C$; therefore DD_1D_2C is the required harmonic range.

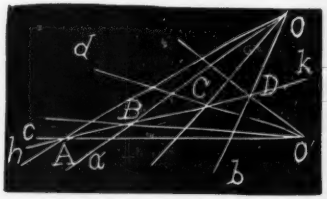
Also solved by C. N. Schmall.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let Oa, Ob, $O'c$, $O'd$ be the four lines; hk the transversal intersecting $O'c$, Oa, $O'd$, Ob in A, B, C, D, respectively.

Let $\alpha=0$ be the equation to Oa; $\beta=0$ the equation to Ob; $\gamma=0$ the equation to $O'c$; $l\alpha-m\beta+n\gamma=0$, the equation to $O'd$.

Then for a harmonic range,



$\alpha-p\beta=0$ is the equation to OC;
$\alpha+p\beta=0$ is the equation to OA;
$\gamma-q(l\alpha-m\beta+n\gamma)=0$ is the equation to $O'B$;
$\gamma+q(l\alpha-m\beta+n\gamma)=0$ is the equation to $O'D$.

$\left\{-\frac{2p\triangle}{b-ap},\ \frac{2\triangle}{b-ap},\ 0\right\}$, are the coordinates of A;

$\left\{\dfrac{2np\triangle}{apm+bn-clp+cm}, \dfrac{2n\triangle}{apm+bn-clp+cm}, -\dfrac{2\triangle(lp-m)}{apm+bn-clp+cm}\right\}$ are the coordinates of C;

$\left\{0, \dfrac{2\triangle(qn-1)}{b(qn-1)+cmq}, \dfrac{2\triangle mq}{b(qn-1)+cmq}\right\}$ are the coordinates of B;

$\left\{\dfrac{2\triangle(qn+1)}{a(qn+1)-qcl}, 0, -\dfrac{2\triangle ql}{a(qn+1)-qcl}\right\}$ are the coordinates of D.

$\alpha mq+\beta pmq-\gamma p(qn-1)=0$, is the line through AB;
$\alpha nql-\beta(qmn+m-lp)+\gamma n(qn+1)=0$, is the line through CD.

Comparing, we get $q=0$ or $q=-(1/n)$, $p=m/l$ or $p=0$. There are no positive values for q when p is positive. When p is positive, $q=0$, etc.

Whatever relations we establish we cannot find p and q both real or both positive.

CALCULUS.

286. Proposed by R. D. CARMICHAEL, Princeton University.

Solve the differential equation

$$[a_0x^3+a_1x^2y+a_2xy^2+(a_0-a_1+a_2)y^3 +a_3x^2+a_4xy+a_5y^2+a_6x+a_7y+a_8]dx +[a_0y^3+a_1xy^2+a_2x^2y+(a_0-a_1+a_2)x^3 +a_3y^2+a_4xy+a_5x^2+a_6y+a_7x+a_8]dy=0.$$

No solution of this problem has been received.

287. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

An object P, being placed beyond the principal focus F of a convex lense, determine its position when its distance PQ, from its image Q, is a minimum.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let u=distance of object from lense, v=distance of image from lense, t=thickness of lense, and r, s=the radii of the first and second surface.

Then $\dfrac{1}{\dfrac{1}{v}+\dfrac{\mu-1}{s}}+\dfrac{1}{\dfrac{1}{u}-\dfrac{\mu-1}{r}}=\dfrac{t}{\mu}$, where μ=index of refraction, or

$$\mu\left(\frac{1}{u}+\frac{1}{v}\right)=\left(\frac{1}{r}-\frac{1}{s}\right)\mu(\mu-1)+t\left(\frac{1}{v}+\frac{\mu-1}{s}\right)\left(\frac{1}{u}-\frac{\mu-1}{r}\right)\ldots(1).$$

$$u+v+t=\text{minimum}\ldots(2).$$

$$\mu\left(\frac{du}{u^2}+\frac{dv}{v^2}\right)=t\left(\frac{1}{u}-\frac{\mu-1}{r}\right)\frac{dv}{v^2}+t\left(\frac{1}{v}+\frac{\mu-1}{s}\right)\frac{du}{u^2}.\quad du+dv=0.$$

$$\therefore\left(\mu-\frac{t(\mu-1)}{r}\right)u^2+tu=\left(\mu+\frac{t(\mu-1)}{s}\right)v^2+tv.$$

The value of v from this equation in (1) gives the value of u.
If t, the thickness of the lense, be neglected, we get $u=\pm v$.
This in $\frac{1}{u}+\frac{1}{v}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)=\frac{1}{f}$, gives $\frac{2}{u}=\frac{1}{f}$, or $u=2f=v$.
Taking the formula $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, $u+v=$minimum, we get

$$\frac{du}{u^2}+\frac{dv}{v^2}=0.\quad du+dv=0.$$

$\therefore u=\pm v$ and $u=2f=v$, as before. The object and image are both twice the focal distance from the lense.

Also solved by C. N. Schmall.

288. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

Find $\int\frac{xdx}{(1+x^3)^{2/3}}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; FRANCIS RUST, M S., Pittsburg, Pa.; and V.M. SPUNAR, Pittsburg, Pa.

Let $1+x^3=x^3z^3$. Then $\frac{xdx}{(1+x^3)^{2/3}}=-\int\frac{dz}{z^3-1}=u.$

Hence, $u=-\frac{1}{3}\int\frac{dz}{z-1}+\frac{1}{3}\int\frac{(z+2)dz}{z^2+z+1}=\frac{1}{6}\log\left[\frac{z^2+z+1}{(z-1)^2}\right]+\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2z+1}{\sqrt{3}}\right]$

$$=\frac{1}{6}\log\left[\frac{(1+x^3)^{2/3}+x(1+x^3)^{1/3}+x^2}{(1+x^3)^{2/3}-2x(1+x^3)^{1/3}+x^2}\right]$$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2(1+x^3)^{1/3}+x}{x\sqrt{3}}\right].$$

Also solved by J. Scheffer, S. G. Barton, and C. N. Schmall.
S. Lefschetz and V. M. Spunar should have been credited for solving 284 and 285 in last issue.

MECHANICS.

237. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

In a naval action an officer observes that in the case of two guns firing, at elevations α and β, respectively, the projectiles of the former fall a feet short of the target while those of the latter land b feet beyond. The initial velocity being the same in both cases, prove that the *true* elevation is

$$\tfrac{1}{2}\sin^{-1}\left[\frac{a\sin 2\beta+b\sin 2\alpha}{a+b}\right].$$

(Suggested by problem 29, page 219, Jeans' *Theoretical Mechanics*.)

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let R=horizontal distance to target, V=initial velocity, g=the gravitational constant, and ϕ=elevation of gun to make the range equal to R.

Then the standard formula for the horizontal range gives for the elevations of ϕ, α, and β:

$$(1)\quad V\sin 2\phi=gR,$$
$$(2)\quad V\sin 2\alpha=g(R-a),$$
$$(3)\quad V\sin 2\beta=g(R+b).$$

Multiplying (3) by a, (2) by b, and adding, we obtain

$$V(a\sin 2\beta+b\sin 2\alpha)=gR(a+b)=V\sin 2\phi(a+b),$$

whence $\phi=\tfrac{1}{2}\sin^{-1}\left[\dfrac{a\sin 2\beta+b\sin 2\alpha}{a+b}\right]$.

Also solved by H. C. Feemster, J. Scheffer, G. B. M. Zeer, S. G. Barton, and J. E. Sanders.

238. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the bounding major-axis being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

Solution by the PROPOSER.

If the area is not in the same vertical plane as the major axis, suppose it is inclined at an angle α. The ellipse projects into an ellipse in a vertical plane having the semi-axes a and $b\cos\alpha$.

Then from 229, pp, 189-190, Vol. XVI, No. 11, we get

$$\bar{x}=\frac{a(16b\cos\alpha\cos\beta+15a\pi\sin\beta)}{4(4b\cos\alpha\cos\beta+3\pi a\sin\beta)},$$

$$\bar{y}=\frac{b\cos\alpha(16\,a\sin\beta+3\pi\,b\cos\alpha\cos\beta)}{4(4b\cos\alpha\cos\beta+3\pi\,a\sin\beta}.$$

The distance of the center of pressure below the surface of the water is

$$\frac{b\cos\alpha\cos\beta(16a\sin\beta+3\pi\,b\cos\alpha\cos\beta)+a\sin\beta(16b\cos\alpha\cos\beta+15a\pi\sin\beta)}{4(4b\cos\alpha\cos\beta+3\pi\,a\sin\beta)}$$

$$=\frac{32ab\sin\beta\cos\beta\cos\alpha+3\pi\,b^2\cos^2\alpha\cos^2\beta+15\pi\,a^2\sin^2\beta}{4(4b\cos\alpha\cos\beta+3\pi\,a\sin\beta)}.$$

Also solved by S. G. Barton.

239. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

A uniform bar of length $2a$ is placed in a sloping position, its lower end on the ground (coefficient of friction being μ), its upper end in the air, the bar being supported by a rough fixed peg (coefficient of friction μ'), against which it rests. If h is the height of the peg from the ground, and if θ be the angle the bar makes with the horizon, when on the point of slipping, prove that θ is to be found from the equation

$$\sin\theta\cos\theta\,[(\mu-\mu')\cos\theta+\sin\theta\,(1+\mu\mu')]=\mu\,h/a.$$

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Let R and R' be the pressures on the ground and peg, respectively, and W the weight. Then resolving the forces vertically and horizontally, and taking moments about the point on the ground, we have the three equations:

$$(1)\quad W=R+R')(\cos\theta+\mu'\sin\theta),$$

$$(2)\quad \mu\,R+R'(\mu'\cos\theta-\sin\theta)=0,$$

$$(3)\quad R'h\csc\theta=Wa\cos\theta,\text{ or }W=\frac{R'h}{a\sin\theta\cos\theta}.$$

Substitute this value of W in (1), multiply the equation by μ and subtract (3) to eliminate R', and we have

$$\mu\cos\theta+\mu\,\mu'\sin\theta-\frac{\mu\,h}{a\sin\theta\cos\theta}-\mu'\cos\theta+\sin\theta=0.$$

Whence, $\sin\theta\cos\theta[(\mu-\mu')\cos\theta+\sin\theta(1+\mu\,\mu')]=\mu(h/a).$

Also solved by G. B. M. Zerr and J. Scheffer.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

164. Proposed by G. J. GRIFFITHS, M. A., in Educational Times (Unsolved).

Prove that the sum of the squares of the reciprocals of all integers which are not divisible by the square of any prime is $15/\pi^2$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and V. M. SPUNAR, Pittsburg, Pa.

A solution of this problem is found on page 211, Vol. XIII, No. 11. Also page 134, Vol. V, No. 5.

The required sum is

$$S=\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\cdots$$

$$=\frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\cdots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots}$$

$$1/\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\cdots=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90}=\sigma_1.$$

$$1/\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}=\sigma.$$

$$S=\frac{\sigma}{\sigma_1}=\frac{\pi^2}{6}/\frac{\pi^4}{90}=\frac{15}{\pi^2}.$$

Also solved by S. Lefschetz.

165. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Factor (if possible), 11, 111, 111, 111.

Solution by the PROPOSER.

The prime factors of 11, 111, 111, 111 are 21649 and 513239. The best method known to me to find them, is to make an extended table of numbers of the form $22n+1$ and after striking out the composite ones use the remaining numbers as trial factors. The table may be extended as required but in any case need not be carried beyond 25000.

PROBLEMS FOR SOLUTION.

ALGEBRA.

335. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

A person has $1800 in notes payable $18 monthly, bearing 10% interest. Find their present value if the interest is payable at the maturity of each note; also present value if interest is payable annually. [An actual business transaction.]

336. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Evaluate the determinant

$$\Delta = \begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \dots & a_n^2 \\ a_2^2, & a_3^2, & a_4^2, & \dots & a_{n+1}^2 \\ a_3^2, & a_4^2, & a_5^2, & \dots & a_{n+2}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n^2, & a_{n+1}^2, & a_{n+2}^2, & \dots & a_{2n}^2 \end{vmatrix}$$

GEOMETRY.

365. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Given the coordinates of the four vertices of the tetrahedron, (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3); (x_4, y_4, z_4): find volume and express it by a determinant.

366. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Astronomy, Manchester, N. H.

Construct a triangle, having given the base, vertical angle, and difference of altitude and difference of other two sides.

367. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The tangents from a point A to a circle are bisected by a line XYZ, which cuts a chord in X and the tangents at its extremities in Y, Z. Show that $XAY=XAZ$, or $XAY=\pi-XAZ$. Also, reciprocate with respect to A.

CALCULUS.

292. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the partial differential equation, $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=axy$.

293. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find the length of the integral curve of the differential equation $(y^2x^{\frac{1}{2}}+2)dx-x^{\frac{1}{2}}dy=0$ between $x_1=1$ and $x_2=8$.

294. Proposed by C. N. SCHMALL, New York City.

Examine the function, $f(x)=\frac{(x-1)(x-2)}{(x-3)}$ and determine why its *minimum* value is *greater* than its maximum.

MECHANICS.

246. Proposed by A. M. HARDING, Adjunct Professor, University of Arkansas, Fayetteville, Ark.

A pentagon $ABCDE$, formed of equal uniform heavy rods connected by smooth joints at their ends, is supported symmetrically in a vertical plane with A uppermost, and AB and AE in contact with two smooth pegs in the same horizontal line. Prove that if the pentagon is regular, the pegs must divide AB and AE each in the ratio $1+\sin(\pi/10):3\sin(\pi/10)$. Jeans' *Theoretical Mechanics*, page 112, number 13.

247. Proposed by C. N. SCHMALL, New York City.

A cylinder of height h and radius r is standing on a horizontal seat in a railway car while the train is getting under way with an acceleration f. Show that the cylinder will not be disturbed if

$$f<\mu g, \text{ and } f<2rg/h,$$

where μ is the coefficient of friction.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

171. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve completely:

$$\begin{aligned} 2x^2-1&=y,\\ 2y^2-1&=z,\\ 2z^2-1&=w,\\ 2w^2-1&=x. \end{aligned}$$

172. Proposed by H. C. FEEMSTER, York, Neb.

Show that $\dfrac{(nr)!}{n!(r!)^n}$ is an integer.

173. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find integral values satisfying the equation,

$$a_1^2+a_2^2+a_3^2+\ldots+a_n^2=d^4.$$

NOTES AND NEWS.

Professor L. E. Dickson, who has been abroad during the past six months, returns to residence at the University of Chicago for the Spring Quarter, beginning April first, 1910. S.

The twenty-sixth regular meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, April 8, 9, 1910, at the Ryerson Physical Laboratory. Arrangements will be made for members present to dine together on Friday evening. S.

Professor H. B. Newson, of the University of Kansas, died suddenly at his home on February 18, at the age of fifty years. Professor Newson had held a full professorship since 1905 and would have succeeded to the headship of the department at the close of the present academic year. He was a graduate of Ohio Wesleyan University and had studied at Johns Hopkins, Heidelberg, and Leipsig. S.

At the annual meeting of the American Federation of teachers of the mathematical and physical sciences in Boston, during the Christmas recess, several important committees made preliminary reports, including the committee on a syllabus in geometry, the committee on college entrance requirements, and various committees under the International Commission on the teaching of mathematics. Two new committees were appointed, one to cooperate with the College Entrance examination board on the best form of logarithmic tables to be used in examinations, and one to consider the question of the publication of a journal devoted exclusively to secondary mathematics. Professor C. R. Mann, of the University of Chicago, was elected president for the ensuing year, and Professor E. R. Smith, of the Brooklyn Polytechnic Institute, secretary. S.

The series of articles which has been running for the past fourteen months in the *New York Independent*, on Great American Universities, was brought to a close in the March issue with a general article making some comparisons of great interest in respect to age, attendance, alumni, libraries, incomes, doctorates, etc. The author of the articles is Edwin E. Slosson, Ph. D. (Chicago). The *Independent* announces a similar series of articles in the near future on Great Foreign Universities. S.

W. B. Fite, assistant professor of mathematics in Cornell University, and H. E. Hawkes, assistant professor of mathematics in Yale University, were appointed full professors of mathematics in Columbia University. Professor C. J. Keyser has been given charge of the mathematical department of Columbia University, succeeding Dean Van Amringe, who retires from active service. M.

Professor Jules Molk, 8 Rue d' Alliance, Nancy, France, invites all mathematicians to mail to him corrections and additions to the published numbers of the great *Encyclopédie des Sciences Mathématiques.* Each volume is to contain "complements" on all the articles of the volume. It is also expected to include a comparative list of the principal technical terms used in the volume, in English, French, German, and Italian. The entire encyclopedia will probably fill more than thirty large volumes. Over 600 pages of the first volume and some parts of the following volumes have appeared. The work is published jointly by Gauthier-Villars of Paris, and B. G. Teubner of Leipzig. M.

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THE TEACHING OF THE CALCULUS.

By W. B. FORD, University of Michigan.

Of the various branches of collegiate mathematics doubtless none have received so much attention on the pedagogical side during recent years as the Calculus. In the newer text books on the subject, for example, unusual care has been taken to produce a high order of pedagogical as well as scientific excellence. The same theme has also played an important rôle of late years in the activities of scientific societies, an instance of which occurs in the title of a recent presidential address read before the American Mathematical Society, which was "The Teaching of the Calculus."*

In view of this wide-spread interest in the subject of the present paper, it is with no little hesitation that the author, in accepting an invitation from the Editors of the MONTHLY to express his own views, undertakes to add anything new or of value to it. However, there are two ideas or views which appear to him deserving of somewhat more prominence than they have yet received, and he gladly takes this occasion to at least give expression to them.

Referring again to the newer texts, we may say that they are distinguished from the old mainly in two particulars. First and foremost, they enliven the calculus by bringing it into closer touch with the physical, this in turn being accomplished through the systematic introduction of problems based upon every day experience or taken from the laboratory. Secondly, the logical flaws, chiefly those relating to infinitesimals which were so prevalent in the texts of a generation ago, have been eliminated through the introduction of closer cut definitions of the terms employed. Needless to say, the effect of these betterments has been widely felt and is constantly making its impress in our colleges today. However, there is doubtless room for still further improvement. The present situation may perhays be described by saying that those pupils, relatively few in number, who really have the power to think may indeed understand the calculus now if they choose, but

*Address of April 27, 1907, by Professor Wm. F. Osgood. See *Bulletin American Mathematical Society*, Vol. 13, pages 449-467.